

A Note on Black Hole Entropy in Loop Quantum Gravity

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Abstract

Several recent results have hinted that black hole thermodynamics in loop quantum gravity simplifies if one chooses an imaginary Barbero-Immirzi parameter $\gamma = i$. This suggests a connection with $SL(2, \mathbb{C})$ or $SL(2, \mathbb{R})$ conformal field theories at the “boundaries” formed by spin network edges intersecting the horizon. I present a bit of background regarding the relevant conformal field theories, along with some speculations about how they might be used to count black hole states. I show, in particular, that a set of unproven but plausible assumptions can lead to a boundary conformal field theory whose density of states matches the Bekenstein-Hawking entropy.

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The computation of black hole entropy [1–3] has been one of the triumphs of loop quantum gravity. The achievement has come at a price, however: the area operator in loop quantum gravity depends on the Barbero-Immirzi parameter γ , and one must tune γ to a rather peculiar value, fixed by an obscure combinatorial problem, to obtain the correct Bekenstein-Hawking entropy [4, 5]. This tuning only has to be done once—if γ is adjusted to fit, say, the Schwarzschild black hole, the theory gives the correct entropy for all black holes—but the physical meaning of this choice remains enigmatic.

Over the past few years, some interesting new hints have emerged. First, it has been observed that in a grand canonical ensemble, the dependence of entropy on γ can be shifted to a chemical potential term for the number of “punctures” where a spin network intersects the horizon [6]. In the presence of suitable holographic matter fields, it may be possible to set this chemical potential to zero [7], and steps have been taken to develop a conformal field theory description [8]. At the same time, it has been noted that if one analytically continues the Barbero-Immirzi parameter to the value $\gamma = i$, certain partition functions automatically give the correct Bekenstein-Hawking entropy [9, 10], and certain spin foam amplitudes acquire a correct imaginary part [11]. The choice $\gamma = \pm i$ is the one for which Ashtekar variables were first introduced [12], and it is a natural one: the loop quantum gravity connection is self-dual at this value, and is the only value for which the Barbero-Ashtekar connection is a diffeomorphism-invariant spacetime gauge field [13, 14]. Unfortunately, it is not clear how to consistently implement reality conditions when γ is complex; at least for practical purposes, a real value seems to be needed.

I do not have the answers to these problems, and it remains possible that a correct thermodynamic treatment could give a value of the entropy that is independent of γ [15]. But in this note I will suggest some ingredients that might be included in the answers, including a few elements of conformal field theory that may not be familiar to most people working in loop quantum gravity. In particular, there are interesting hints that coset constructions, Liouville theory, and holomorphic Chern-Simons theories may be relevant.

1. Chern-Simons and WZNW theory

A question about a black hole in quantum gravity is a question about conditional probability: one must impose the condition that the desired black hole is present. One way to do so is to demand that a hypersurface Δ be a horizon. Such a horizon is not a physical boundary, but it is a place where one is imposing “boundary conditions,” and as a consequence, the Einstein-Hilbert action acquires an added boundary term. In the first-order formalism of loop quantum gravity, this term is a three-dimensional Chern-Simons action [2, 19, 20],

$$I_{CS} = \frac{k}{4\pi} \int_{\Delta} \text{Tr} \left\{ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\}, \quad (1.1)$$

where $A = A_{\mu}^a T_a dx^{\mu}$ is a Lie-algebra-valued connection one-form.* In loop quantum gravity, it is usual to start with a self-dual $\text{SL}(2, \mathbb{C})$ connection, making (1.1) a holomorphic $\text{SL}(2, \mathbb{C})$ Chern-Simons action. (The general $\text{SL}(2, \mathbb{C})$ Chern-Simons action contains a second term involving the complex conjugate \bar{A} .) From the point of view of Chern-Simons theory, this choice—and thus the choice $\gamma = i$ —is special: holomorphic Chern-Simons actions are expected to have exceptional properties, but they are also exceptionally poorly understood [21].

*The appearance of a Chern-Simons boundary action was first suggested in a slightly different context by Smolin [16, 17]. An related $\text{SL}(2, \mathbb{C})$ Chern-Simons action also appears in certain spin foam amplitudes for (3+1)-dimensional gravity with a positive cosmological constant [18], although the connection to black hole horizons is not yet clear.

Chern-Simons theory is intimately related to a two-dimensional conformal field theory, Wess-Zumino-Novikov-Witten (WZNW, or WZW) theory [22]. To see this, consider the case that Δ itself has a boundary, which may again be merely a “place where one imposes boundary conditions.” The action (1.1) must now itself acquire a boundary term: under a variation of A , we have

$$\delta I_{CS} = \frac{k}{2\pi} \int_{\Delta} \text{Tr} [\delta A (dA + A \wedge A)] - \frac{k}{4\pi} \int_{\partial\Delta} \text{Tr} (A \wedge \delta A),$$

and we must modify the action to cancel the boundary variation. To simplify comparison with the literature, choose complex coordinates $\{z, \bar{z}\}$ on $\partial\Delta$, and fix A_z and the boundary.[†] The required boundary term is then

$$I_{bdry}[A] = \frac{k}{4\pi} \int_{\partial\Delta} \text{Tr} A_z A_{\bar{z}}. \quad (1.2)$$

We now make the crucial observation that the total action (1.1–1.2) is not gauge invariant at the boundary [23, 24]. If we write A as a gauge-fixed value \bar{A} and a gauge transformation $g(z, \bar{z})$,

$$A = g^{-1} dg + g^{-1} \bar{A} g, \quad (1.3)$$

we find that

$$(I_{CS} + I_{bdry})[A] = (I_{CS} + I_{bdry})[\bar{A}] + k I_{WZW}^+[g^{-1}, \bar{A}], \quad (1.4)$$

with

$$I_{WZW}^+[g^{-1}, \bar{A}_z] = \frac{1}{4\pi} \int_{\partial\Delta} \text{Tr} (g^{-1} \partial_z g g^{-1} \partial_{\bar{z}} g - 2g^{-1} \partial_z g \bar{A}_z) + \frac{1}{12\pi} \int_{\Delta} \text{Tr} (g^{-1} dg)^3. \quad (1.5)$$

This is precisely chiral WZNW action for a field g coupled to a background gauge potential \bar{A}_z .

The gauge transformation g has thus become dynamical at the boundary. In a reasonable sense, in fact, it comprises most of the degrees of freedom: a Chern-Simons theory on a closed manifold is a topological field theory with only finitely many degrees of freedom, while a WZNW model is a true field theory, albeit in one dimension less. The boundary degrees of freedom have a physical interpretation as “would-be gauge degrees of freedom” [25], degrees of freedom that are only physical because the boundary conditions break the gauge invariance of the bulk. Their presence is necessary for consistency—for instance, if one “glues” two manifolds along a common boundary, the path integrals for Chern-Simons theories on each half combine correctly only if one includes the boundary WZNW model [26].

For black holes, two types of boundaries of Δ are of particular interest. Suppose that Δ initially has the usual horizon topology $\mathbb{R} \times S^2$. If we cut a “hole” in a spherical cross-section of Δ , the resulting boundary will be a cylinder $\partial\Delta \approx \mathbb{R} \times S^1$. The dynamical variable in (1.5) will be a group-valued field $g(\phi, v)$, and (1.5) will describe a WZNW model on a cylinder. This is a well-understood quantum theory [27], which can be quantized, for instance, in terms of loop groups [28].

Suppose, on the other hand, that we shrink the “hole” to a point p , a “puncture.” The boundary $\mathbb{R} \times \{p\}$ is now a line, which can be interpreted in the Chern-Simons theory as a Wilson line. The field g no longer has any spatial dependence; the only remaining information is the holonomy around the one-dimensional boundary, and the quantum theory shrinks to a theory of the conformal blocks of the WZNW model [22, 28].

[†]This choice simplifies notation, but can be easily generalized. For each value of the Lie algebra index a , A_z^a and $A_{\bar{z}}^a$ are canonically conjugate; one can fix any linear combination of these at the boundary and leave the conjugate component free.

2. Virasoro algebras and central charge

The WZNW model (1.5) is a two-dimensional conformal field theory. Such theories have a very powerful infinite-dimensional symmetry group, which determines much of their behavior. The metric for a two-dimensional manifold can always be written locally as

$$ds^2 = 2g_{z\bar{z}}dzd\bar{z}$$

where the complex coordinates $\{z, \bar{z}\}$ were introduced in the preceding section. The holomorphic and antiholomorphic diffeomorphisms $z \rightarrow z + \xi(z)$, $\bar{z} \rightarrow \bar{z} + \bar{\xi}(\bar{z})$ rescale the metric, and a conformal field theory is invariant under such rescalings. The canonical generators of this symmetry, denoted $L[\xi]$ and $\bar{L}[\bar{\xi}]$, satisfy a Virasoro algebra [29],

$$[L[\xi], L[\eta]] = L[\eta\xi' - \xi\eta'] + \frac{c}{48\pi} \int dz (\eta'\xi'' - \xi'\eta'') \quad (2.1a)$$

$$[\bar{L}[\bar{\xi}], \bar{L}[\bar{\eta}]] = \bar{L}[\bar{\eta}\bar{\xi}' - \bar{\xi}\bar{\eta}'] + \frac{\bar{c}}{48\pi} \int d\bar{z} (\bar{\eta}'\bar{\xi}'' - \bar{\xi}'\bar{\eta}'') \quad (2.1b)$$

$$[L[\xi], \bar{L}[\bar{\eta}]] = 0. \quad (2.1c)$$

The first terms on the right-hand sides of (2.1a–2.1b) give the usual algebra of diffeomorphisms. The remaining terms provide a unique central extension, fixed by the values of the two constants c and \bar{c} , the central charges. As usual, the zero modes Δ_0 and $\bar{\Delta}_0$ of L_0 and \bar{L}_0 are conserved quantities, the “conformal weights,” which can be viewed as linear combinations of mass and angular momentum.

Remarkably, Cardy has shown that with a few mild restrictions, the asymptotic density of states of any two-dimensional conformal field theory is almost completely determined by the symmetries [30, 31]. Let Δ_{min} and $\bar{\Delta}_{min}$ be the lowest eigenvalues of L_0 and \bar{L}_0 (usually but not always zero), and define

$$c_{eff} = c - 24\Delta_{min}, \quad \bar{c}_{eff} = \bar{c} - 24\bar{\Delta}_{min}. \quad (2.2)$$

Then the density of states ρ in a microcanonical ensemble, at fixed eigenvalues Δ and $\bar{\Delta}$ of L_0 and \bar{L}_0 , behaves as [32]

$$\ln \rho(\Delta) \sim 2\pi \sqrt{\frac{c_{eff}\Delta}{6}}, \quad \ln \bar{\rho}(\bar{\Delta}) \sim 2\pi \sqrt{\frac{\bar{c}_{eff}\bar{\Delta}}{6}} \quad (2.3)$$

and in a canonical ensemble, at fixed temperature T , as [33]

$$\ln \rho(T) \sim \frac{\pi^2}{3} c_{eff} T, \quad \ln \bar{\rho}(T) \sim \frac{\pi^2}{3} \bar{c}_{eff} T \quad (2.4)$$

where T is a dimensionless temperature, determined by the periodicity of a dimensionless conformal time. (Under some circumstances, the “left temperature” T and “right temperature” \bar{T} may also differ.) The entropy is thus determined by a few parameters, independent of the details of the theory.

There is one remaining subtlety: a two-dimensional conformal field theory may be defined on the complex plane or on a cylinder. The two are related by the transformation $w = \ln z$, but because of the conformal anomaly, the Virasoro generators do not quite transform as tensors [29]. Rather,

$$\Delta_{cyl} = \Delta_{plane} - \frac{c}{24}, \quad \bar{\Delta}_{cyl} = \bar{\Delta}_{plane} - \frac{\bar{c}}{24}. \quad (2.5)$$

The conformal weights Δ and $\bar{\Delta}$ appearing in (2.3)–(2.4) are those for the cylinder, as may be checked from the derivation in [32].

3. $\text{SL}(2)$ connections, coset constructions, and Liouville theory

In the first order formulation of general relativity, in addition to the diffeomorphisms, the fundamental symmetries include local Lorentz transformations, which take their values in $\text{SL}(2, \mathbb{C})$. In the usual approach to loop quantum gravity, one gauge-fixes the latter to $\text{SU}(2)$, the little group of the timelike normal to a spatial slice. The boundary Chern-Simons theory is then an $\text{SU}(2)$ theory—perhaps further reducible to $\text{U}(1)$ [3]—with a real Barbero-Immirzi parameter. Both the area of the horizon and the dimension of the Chern-Simons Hilbert space are determined by the spins carried by spin network edges crossing the horizon. Determining the relationship between the two—and thus the Bekenstein-Hawking entropy—becomes a combinatorial problem; see, e.g., [34] for a careful treatment in the $\text{SU}(2)$ case.

As noted above, though, this choice requires an awkward fine-tuning of the Barbero-Immirzi parameter, and there are hints that the picture may simplify if one can instead choose $\gamma = i$. This requires a complex connection, and suggests that we should look at the full gauge group of self-dual connections, complexified $\text{SU}(2)$, which is isomorphic to $\text{SL}(2, \mathbb{C})$. This is the group that seems most natural in light of the linear simplicity constraints of spin foam theory [35]

$$K^i = \gamma L^i \quad (3.1)$$

with $\gamma = i$, where the K^i and L^i are generators of “internal” boosts and rotations.

$\text{SL}(2, \mathbb{C})$ may be too large a group, however. The analytic continuation to imaginary γ in [9, 10] is really a continuation to an $\text{SL}(2, \mathbb{R})$ (or, equivalently, $\text{SU}(1, 1)$) algebra. Frodden et al. [9] argue that this subgroup is picked out by a reality condition for the area; it is also the little group for a unit normal to a stretched horizon.[‡]

For now, both of these possibilities seem worth considering. Of course, this is not trivial—in neither case do we know how to quantize the theory, except in some sense as an analytic continuation from real γ . Still, it is worth seeing where these alternatives might take us. We might at first expect that such choices should be irrelevant: unless there is an anomaly, it should not matter how we gauge-fix a symmetry. But as noted earlier, boundary conditions can break the symmetry, elevating “would-be gauge transformations” to true degrees of freedom. If these are indeed relevant degrees of freedom, the choice of gauge group and the pattern of symmetry-breaking could be crucial.

From section 1, we now expect an $\text{SL}(2, \mathbb{C})$ or $\text{SL}(2, \mathbb{R})$ Chern-Simons theory on the horizon. Neither of these is a very well-understood theory (though see [21, 38–40]). Fortunately, though, there is a potential simplification: we have not yet exhausted our use of boundary conditions.

An isolated horizon [19] is characterized by a null normal ℓ^a satisfying the condition $\ell^b \nabla_b \ell^a = \kappa \ell^a$ on Δ . Following [42], let us choose a constant “internal” vector ℓ^I such that $\ell^a = e^a_I \ell^I$. The spin connection ω must then satisfy $\ell^a \omega_a^I{}_J \ell^J = \kappa \ell^I$ at the horizon, and it may be checked that this implies

$$A_v^I{}_J \ell^J \propto \ell^I \quad (3.2)$$

for the full self-dual connection. From (1.3), this means that certain components of the WZNW current

$$(J_v)^I{}_+ = k (g^{-1} \partial_v g + g^{-1} \bar{A}_v g)^I{}_+$$

are fixed.

Modifications of this type—in which a current corresponding to a subgroup of the gauge group is held fixed—have been the subject of extensive study. The resulting conformal field theories are known as coset

[‡]There are also arguments for a smaller $\text{ISO}(2)$ or $\text{ISO}(1, 1)$ gauge group [36, 37].

models [29, 41]. In our case, the relevant coset is $SL(2, \mathbb{C})/E(2)$, where $E(2)$ is the little group of the null vector ℓ^A , the (complexified) Euclidean group. This particular coset model has not been investigated in great detail, but Ghezelbash [43] has argued that the resulting conformal field theory is a Liouville theory

$$I_{\text{Liou}} = \frac{k}{8\pi} \int d^2x \sqrt{g} \left(\frac{1}{2} g^{ab} \partial_a \varphi \partial_b \varphi + \frac{1}{2} \varphi R + \lambda e^\varphi \right) \quad (3.3)$$

with a central charge $c = 6k$. This result is semiclassical, and the setting is not quite right for loop quantum gravity, since the connection in [43] is not self-dual. But the result is at least suggestive. Coset models based on $SL(2, \mathbb{R})$ have been considered much more carefully, and they also, much less ambiguously, reduce to Liouville theory with the same central charge [44–50].

Liouville theory is a well-known conformal field theory, which has been widely studied [51, 52] but is still not completely understood. The quantum states come in two sets, commonly called the “normalizable” and “nonnormalizable” or “microscopic” and “macroscopic” sectors. The states in the normalizable sector have conformal weights that are bounded below by finite values,

$$\Delta_{\min} = \bar{\Delta}_{\min} = \frac{c-1}{24} ,$$

so by (2.2), the effective central charges are the same as those of a free scalar field, $c_{\text{eff}} = \bar{c}_{\text{eff}} = 1$. The quantization of this sector is relatively well understood. The nonnormalizable sector, on the other hand, has conformal weights that go down to zero, so the effective central charge is large for large k , but the quantization of this sector is much more poorly understood (though see [53] for one interesting attempt). It is suggestive that the normalizable sector corresponds classically to black-hole-like geometries, while the more mysterious nonnormalizable sector corresponds to point-particle-like geometries of the sort that arise in descriptions of spin network intersections with a black hole horizon [51, 53, 54].

4. The three-dimensional gravity connection

We may be able to learn more about the boundary action (1.1) by taking advantage of an accidental relationship with three-dimensional gravity. In three-dimensional asymptotically anti-de Sitter space with a cosmological constant $\Lambda = -1/\ell^2$, the triad $e^a = e_\mu^a dx^\mu$ and spin connection $\omega^a = \frac{1}{2} \epsilon^{abc} \omega_{\mu ab} dx^\mu$ can be combined as

$$A^{(\pm)a} = \omega^a \pm \frac{\sqrt{-\sigma}}{\ell} e^a \quad (4.1)$$

where σ is the signature of the metric (+1 for Riemannian, −1 for Lorentzian). As noted by Achúcarro and Townsend [55] and elaborated by Witten [56], the Einstein-Hilbert action then becomes a difference of Chern-Simons actions for $A^{(\pm)}$, with coupling constants $k = \ell \sqrt{-\sigma}/4G_3$. In particular, the boundary action (1.1) with gauge group $SL(2, \mathbb{C})$ is equivalent to the action for three-dimensional Euclidean anti-de Sitter gravity. For the self-dual (or “holomorphic”) case, the action includes an extra “exotic” term [56], but the field equations are unaffected. Similarly, the boundary action with gauge group $SL(2, \mathbb{R})$ is equivalent to a chiral half of the action for three-dimensional Lorentzian anti-de Sitter gravity.

Now, Brown and Henneaux showed long ago [57] that the asymptotic symmetries of three-dimensional anti-de Sitter gravity are conformal symmetries, described by a Virasoro algebra with a central charge

$$c = \bar{c} = \frac{3\ell}{2G_3} = \frac{6k}{\sqrt{-\sigma}}. \quad (4.2)$$

While the microscopic degrees of freedom are not very well understood [58], we have very good reasons to believe this result, especially since the Cardy formula then gives the correct entropy for the three-dimensional BTZ black hole.

To be sure, the physical settings for conformal symmetry differ for the two theories. In three-dimensional gravity, the relevant symmetry group appears at asymptotically anti-de Sitter boundaries at infinity, while in loop quantum gravity we are interested in boundaries in the form of punctures or holes at the event horizon. But the two geometries are related by conformal transformations, and the relevant conformal weights in three-dimensional gravity are determined by holonomies of the connection, which are invariant under such transformations. If we can better understand the conformal weights in loop quantum gravity, it should be possible to translate these into the setting of three-dimensional gravity.

An alternative connection to (2+1)-dimensional gravity may also be useful. In addition to describing three-dimensional anti-de Sitter gravity with a Riemannian signature metric, an $\text{SL}(2, \mathbb{C})$ Chern-Simons action describes (2+1)-dimensional de Sitter gravity with Lorentzian signature. The counting of states in this theory is not quite as well understood, but there is very strong evidence that it can again be described in terms of a two-dimensional conformal field theory. In [60], it is shown that this equivalence yields the correct black hole entropy for loop quantum gravity in a rather straightforward way.

5. Entropy from the canonical Cardy formula

Let us suppose for now that the horizon boundary term in loop quantum gravity can be related to a Liouville theory, as suggested in section 3, and that the effective central charge is given by the full central charge $c = 6k$ (that is, that we include the nonnormalizable sector). Alternatively, we may assume that the connection to three-dimensional gravity described in section 4 gives the correct central charge. To determine the significance of this central charge in the (3+1)-dimensional setting, we need the coefficient k of the Chern-Simons action (1.1). This is given in refs. [2, 3, 19]:

$$k = \frac{iA_\Delta}{8\pi G}, \quad (5.1)$$

where A_Δ is the prescribed area of the horizon. The appearance of an imaginary coupling constant is slightly peculiar, but it is expected for a holomorphic Chern-Simons theory, where it can be traced to the factor of $\sqrt{-\sigma}$ in (4.1). The factor of i disappears in the central charge (4.2), and one can again find a relationship with Liouville theory with central charge $6k$ [59]. It is interesting to note that this expression gives one-half of the central charge obtained by looking at boundary terms in the generators of diffeomorphisms [61, 62].

We also need the dimensionless temperature T . The natural choice is the “geometric temperature” [63], the temperature for which the horizon state is a KMS state,

$$T = \frac{1}{2\pi \left(1 - \frac{1}{k}\right)} \approx \frac{1}{2\pi}. \quad (5.2)$$

More simply, in the large k limit for which our analysis might be trusted, this is simply the local temperature seen by a preferred quasilocal observer a proper distance ℓ from the horizon, scaled by ℓ to become dimensionless [64].

We can now insert these values into the canonical Cardy formula (2.2). We obtain a contribution of

$$S = \frac{\pi^2}{3} c_{\text{eff}} T = \frac{A_\Delta}{8G}, \quad (5.3)$$

one-half of the Bekenstein-Hawking entropy, from each boundary of Δ upon which our conformal field theory appears.

This result is somewhat less informative than one might hope. It requires that the horizon Δ have a boundary—the WZNW model needs some two-manifold to live on—but the result depends on no particular characteristic of that boundary. Rather, we learn that whatever the boundary, equilibrium at the temperature (5.2) requires that the conformal field theory be excited in such a way that the density of states is given by (5.3).

In an eternal black hole, for instance, one possibility is to consider the bifurcation sphere to be an initial boundary of Δ , with a second boundary at the infinite future. This is a somewhat unusual picture, since the boundaries are spacelike, but Chern-Simons theory is a topological field theory, and may not care about the distinction between spacelike and timelike boundaries. In this case, one gets a contribution (5.3) occurring twice, giving the full Bekenstein-Hawking entropy. Another possibility, currently under investigation, is that the relationship with three-dimensional gravity may permit a formal identification of the entropy (5.3) with that of one of the two asymptotic regions of a three-dimensional black hole.[§]

6. Counting states with the microcanonical Cardy formula

While the canonical result for black hole entropy is interesting, it would be nice to know more about the actual physical nature of the states. To do so is more difficult, and may require one added assumption.

In the usual approach to black hole entropy in loop quantum gravity, horizon states are associated with punctures where spin network edges cross the horizon. Following [8], let us enlarge these punctures to “holes” with finite, although arbitrarily small, boundaries. As discussed in section 1, the boundary states then become the states of a conformal field theory on a cylinder.

As noted in section 4, we do not really know the relevant states from first principles. We do know that spin network punctures depend on holonomies, and in the three-dimensional gravity model these holonomies determine the conformal weights. From past work involving analytic continuation of the partition function, the states that seem to be relevant to counting spin network edges [9, 10] are classified by the unitary series of $\mathrm{SL}(2, \mathbb{C})$ representations, with $j = \frac{1}{2} + is$. These occur in the relevant WZNW models (see section 4 of [39], sections 1–2 of [40], or [65]), and have conformal weights

$$\Delta_j = -\frac{j(j-1)}{k-2}. \quad (6.1)$$

Similarly, Liouville theory contains states with weights [53, 66]

$$\Delta_j = \frac{c-1}{24} - \frac{j(j-1)}{k-2}. \quad (6.2)$$

(These can also be obtained from [51] by the substitution $j \rightarrow j - \frac{\bar{k}+2}{2}$.) The weights (6.1) and (6.2) differ by a constant, but the Liouville values were obtained for the theory on the plane; as noted in section 2, one must first subtract $c/24$ before using the Cardy formula.

If we now insert these conformal weights into the microcanonical Cardy formula (2.3), and again assume a central charge $6k$, we obtain a contribution of

$$S \approx 2\pi \sum_{\text{punctures}} \sqrt{-j(j-1)}. \quad (6.3)$$

[§]One might also look for a factor of two by considering a nonchiral WZNW model, with contributions from left- and right-moving modes. But the self-dual/holomorphic connection formalism leads much more naturally to a chiral model.

(The square root is real for $j = \frac{1}{2} + is$.) This is almost precisely the standard formula for area in loop quantum gravity [67, 68]. In fact, following the analytic continuation arguments of [9, 10], the right-hand side can be interpreted as exactly one quarter of the horizon area in loop quantum gravity, yielding the correct Bekenstein-Hawking entropy. A similar expression for the area can be obtained from the Alexandrov’s Lorentz-covariant quantization [14, 69], although there the area acquires an extra factor of 2, becoming $16\pi G\sqrt{s^2 + 1/4}$ and making (6.3) one-half of the Bekenstein-Hawking entropy.

7. A few inconclusive conclusions

To conclude, let us first see how the program sketched here contrasts with the standard loop quantum gravity approach to black hole thermodynamics. I believe there are three essential differences: the choice of gauge group, the treatment of spin network intersections with the horizon, and the role of combinatorics.

The first of these—the shift from $SU(2)$ to $SL(2, \mathbb{C})$ or $SL(2, \mathbb{R})$ —is suggested by a number of recent results in which the Barbero-Immirzi parameter is analytically continued to its self-dual value $\gamma = i$. This step is obviously problematic, since we do not yet really understand the quantization of the theory when γ is not real, but perhaps the lesson is that we need to develop a better understanding.

The second—the treatment of spin network intersections with the horizon as “holes” rather than “punctures”—is less clear. The usual identification of punctures comes in several steps. First, isolated horizon boundary conditions require that at a fixed time t , the fields at the horizon satisfy

$$F^{KL} \triangleq -\frac{\pi(1 - \gamma^2)}{A_H} \Sigma^{KL} \quad (7.1)$$

where F^{KL} is the curvature of the self-dual connection A^{KL} , $\Sigma^{KL} = \frac{1}{2}\epsilon^{KL}{}_{MN}e^M \wedge e^N$, and the differential forms are pulled back to the horizon. When the right-hand side of (7.1) is evaluated on a bulk spin network state, it gives a sum of delta functions, one at each point where a spin network edge intersects the horizon, each multiplied by a generator J^{KL} of the gauge group. Eqn. (7.1) is then a condition on the curvature F , which can be recognized as precisely the constraint equation for a Chern-Simons theory at time t with a set of Wilson lines carrying the same representations as the spin network edges. The WZNW state is thus induced by Wilson lines rather than finite boundaries.[¶]

Observe, however, that Σ^{KL} is not, strictly speaking, a well-defined operator in loop quantum gravity: it is distributional, and one should really deal only with integrals $\hat{\Sigma}_T$ over surfaces T [3, 68]. Here, the only relevant surfaces are portions of the horizon itself. Classically, of course, the horizon is a continuous surface, but quantum mechanically this is not at all clear. Since only integrals over “plaquettes” are well-defined in the quantum theory, one might argue that (7.1) should hold only in integrated form.

If this is the case, it is no longer obvious whether spin network intersections should be treated as punctures or as holes. One way to pose this question, as described in section 1, is to ask whether any residual gauge transformations act on loops surrounding an intersection. It is not clear to me whether the theory as currently formulated is capable of answering this question.

Finally, note that at least in section 6, the entropy (6.3) is that of a *fixed* set of punctures, and does not include combinatorial factors from the different ways spin network punctures could give the same horizon area. In contrast, in the standard approach of [2], the entropy arises entirely from such combinatorics.

[¶]Note that the Wilson lines are *not* the same as the spin network edges: the Wilson lines live inside the three-manifold Δ , while the edges merely intersect Δ . One might visualize a Wilson line in terms of spin foams, as a *history* of an intersection of an edge with the horizon.

Indeed, the strange standard value of the Barbero-Immirzi parameter can be traced to the properties of this combinatorial problem.

While the counting of states in a conformal field theory is also a combinatorial problem, it is a rather different one, essentially the determination of the number of partitions of a large integer [32]. This is one of the few such problems whose answer is a natural exponential (the Cardy formula), leading to a simple dependence on the Barbero-Immirzi parameter. But if the ideas presented here are correct, one must understand why the usual combinatorial factor is either absent or subleading. One possibility [70] is that different choices of punctures correspond to genuinely different macroscopic states, since the bulk geometries that can be consistently attached are also different in the neighborhood of the punctures. This interpretation gains support from the successes of canonical approaches that treat the number of punctures as a new thermodynamic state variable [6]. In essence, this becomes question of how much coarse-graining is required to define the entropy of a black hole.

Even if these problems can be addressed, a good deal remains to be done. It is not obvious which Chern-Simons theory is relevant: an $SL(2, \mathbb{R})$ gauge group would simplify life, since many more results are known, but I do not know how to pick out a particular $SL(2, \mathbb{R})$ factor from the full self-dual $SL(2, \mathbb{C})$ action. If the $SL(2, \mathbb{C})$ Chern-Simons theory is indeed the correct one, many questions remain open, from the basic properties of a holomorphic Chern-Simons theory to the inadequately understood coset constructions. If the end result is a Liouville theory, much is still to be understood about the nonnormalizable sector, which must be included for the Cardy formula to give the correct entropy. It has been argued in a different context that Liouville theory should be viewed as an effective description of more fundamental degrees of freedom [71]; this issue remains unresolved.

Finally, it would be valuable to understand the relationship between the approach described here and the more general attempts, summarized in [72], to understand the entropy of arbitrary black holes. Both rely on two-dimensional conformal field theory, but at first sight the field theories are quite different: the WZNW models and Liouville theory I have described here live on the “ φ - v cylinder,” while the models considered in [72] live on the “ r - v plane.” A similar mismatch occurs in the attempt to define a local temperature at the horizon in loop quantum gravity [63]. There, the simplicity constraints (3.1) play a crucial role by connecting spatial rotations to boosts, thus tying together symmetries in different planes. Pranzetti has suggested that a similar mechanism may be at work here [73], but this, like much else in this paper, remains speculative.

The hints I have presented here thus raise as many questions as they offer answers. I believe, though, that some of these are at least the right questions.

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